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Surfaces Foliated by Planar Geodesics:
A Model for Curved Wood Design

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Abstract

Surfaces foliated by planar geodesics are a natural model for surfaces made from wood strips. We outline how to construct all solutions, and produce non-trivial examples, such as a wood-strip Klein bottle.

Introduction

The construction of curved surfaces by bending flat strips of wood and placing them side by side has a long history. In some settings, such as in the design of furniture or art-works, a *curved* surface is wanted for aesthetic reasons. In other applications, such as ship-building or air-craft design (Figure 1, left), the need for a curved surface is dictated by practical considerations, e.g., aerodynamic efficiency.

Mathematically, any regular surface can be approximated by such a wood-strip construction, simply by choosing some 1-parameter family of *geodesics* (Figure 1) that covers the surface (several patches might be needed). A geodesic is a curve that only bends in the direction perpendicular to the surface. Thus, laying a wooden strip along a geodesic means that the normal vector to the strip is *parallel* to the normal vector to the surface being modeled. In other words, the strips are tangent to the surface – at least along their centerlines.

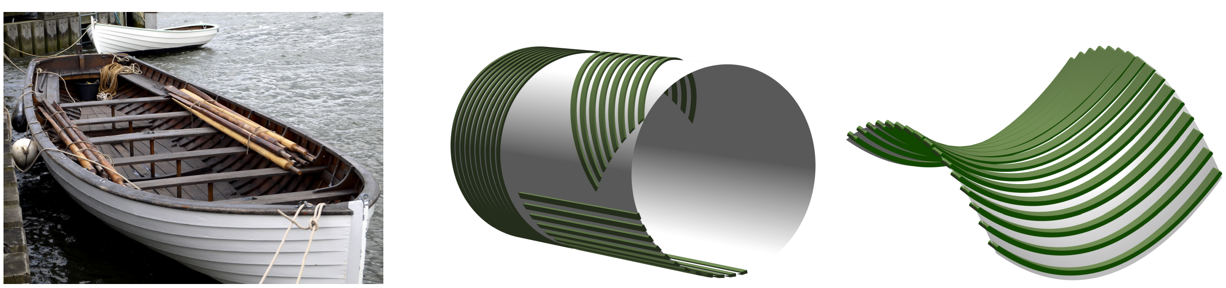

Figure 1 : *Left: curved wooden boats. Middle: geodesics on a cylinder. Right: non-geodesic curves.*

For practical assembly, the bending of the strips must be achieved by fixing the positions of various points with, for example cross-beams. If the curve that the strip follows is *planar*, then the shape of the curve between these fixed points is predictable, independent of the material or thickness of the strip, provided no plastic deformation occurs. Moreover, a wood strip can follow the curve with no twisting. It is desirable, therefore, to find a 1-parameter family of *planar geodesics*. Such a family cannot be expected to exist on an arbitrary surface. Therefore, the problem of mathematically designing a surface for wood strip assembly involves first understanding how to generate the right kind of surface, and this is the main purpose of this article. Note that the considerations above apply equally well if we replace wood with paper or cardboard, both popular media for artistic expression.

Surfaces Foliated by Planar Geodesics

A family of smoothly varying curves that sweeps out a surface (such as one of the families of coordinate lines in a parameterization) is called a *foliation*. We are interested in the case that each curve is a planar geodesic.

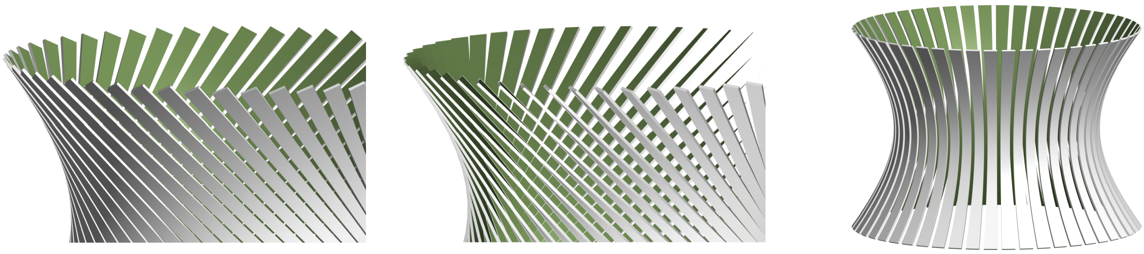

Figure 2 : *Hyperboloid. Left, Middle: as a ruled surface. Right: as a surface of revolution.*

The most obvious example is a *ruled surface*, i.e., a surface swept out by a family of straight lines, such as the hyperboloid in Figure 2. However, ruled surfaces are an exceptional case because a given line lies in infinitely many planes, unlike a typical plane curve. Moreover, a line does not have a unique normal vector, so there are many different ways to lay a wood strip along it: two choices are shown in Figure 2. Finally, and most importantly, the surface normal is not usually constant along a ruling: thus, it is not possible to lay a wood strip along the ruling in such a way that the normal of the wood strip is the same as the normal to the underlying surface without twisting the wood. The exception to this is a *developable surface* defined to be a ruled surface in which the surface normal is constant along each of the rulings. These are the only ruled surfaces that satisfy the following

Definition. *A* planar geodesic foliation *(PGF) is a surface foliated by planar geodesic curves such that, along each of these planar curves, the normal to the surface is parallel to the plane containing the curve.*

Rotational Surfaces: Although not developable, the hyperboloid nevertheless qualifies as a PGF, because it happens to be an instance of a second well known type of surface foliated by planar geodesics: a *surface of revolution*. Take a *profile curve* $\gamma_0(u) = (f(u), 0, g(u))$ in the xz-plane, and then rotate this plane about the z-axis by angle v to get a new plane Π_v. As v varies, the curve sweeps out a *surface of revolution* $\mathbf{x}(u, v) = (\cos(v)f(u), \sin(v)f(u), g(u))$ (Figure 3). As is well-known, the plane Π_v is orthogonal to the surface of revolution, and this implies that each iso-curve $\gamma_{v_0}(u) = (\cos(v_0)f(u), \sin(v_0)f(u), g(u))$ is a geodesic.

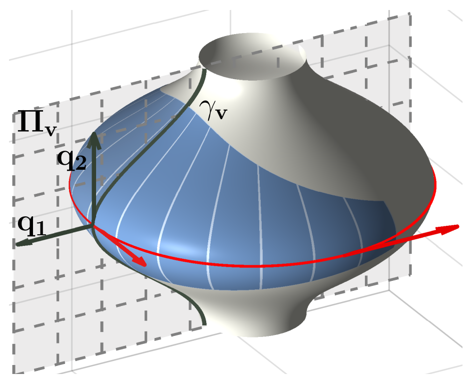

Figure 3 : *A* subset *(striped) of a rotational surface is also foliated by planar geodesics.*

Monge Surfaces: A surface of revolution is obtained from a plane curve by a perpendicular sweep of the plane along a circle. If we replace the circle with an arbitrary space curve, the *spine curve*, that is always perpendicular to the moving plane, we have a *Monge surface* ([4], §XXV). Again, each iso-curve $\gamma_v(u) = \mathbf{x}(u, v)$ is a planar geodesic. (See Figure 4). Note that both developable surfaces and surfaces of revolution are also Monge surfaces.

Constructing a Monge surface: Given a plane Π (which contains the profile curve γ_0), and a candidate differentiable space curve $\mathbf{r}(v)$ for the spine curve, with $\mathbf{r}'(v_0)$ perpendicular to Π, one can take advantage of the so-called *minimal rotating frame* (MRF) defined by R. Bishop [1]. Take an orthonormal basis Q_1, Q_2 for Π: then there is a unique orthonormal frame field $(\mathbf{t}(v), \mathbf{q}_1(v), \mathbf{q}_2(v))$ that evaluates at v_0 to $(\mathbf{r}'(v_0)/|\mathbf{r}'(v_0)|, Q_1, Q_2)$, and such that the derivatives of the frame vectors have components only in the direction $\mathbf{r}'(v)$. The plane $\Pi_v = \mathrm{span}(\mathbf{q}_1(v), \mathbf{q}_2(v))$ remains perpendicular to the spine curve for all v, and a Monge surface is swept out by γ_0 as the plane moves. The frame is calculated by solving an ODE (see [2]).

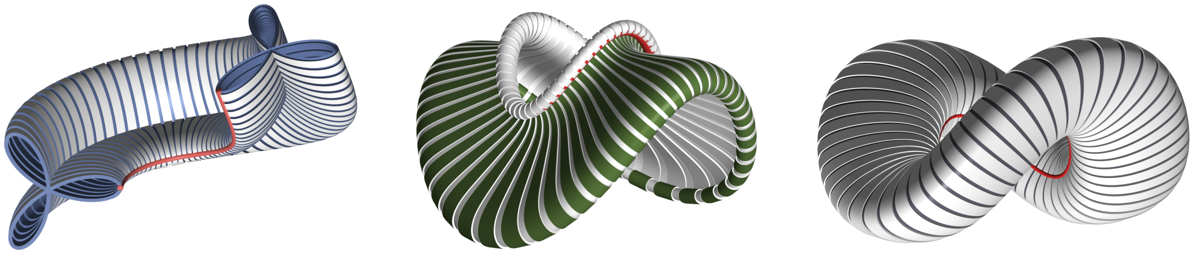

Figure 4: *Monge surfaces: the middle and right surface have the same spine curve (red) but different profile curves.*

It follows from the above that a Monge surface can be parameterized as:

$$\mathbf{x}(u,v) = \mathbf{r}(v) + x(u)\,\mathbf{q}_1(v) + y(u)\,\mathbf{q}_2(v)\,, \tag{1}$$

with $\mathbf{r}$ a space curve with MRF ($\mathbf{t} = \mathbf{r}'/|\mathbf{r}'|, \mathbf{q}_1, \mathbf{q}_2$), and each iso-curve $\gamma_v(u) = \mathbf{x}(u,v)$ is then a geodesic.

How to Produce All Planar Geodesic Foliations

From the point of view of designing with wood strips, it's obvious to ask now whether there are any other PGF's, besides Monge surfaces. To better understand this question, let's consider one more application of PGF's. In architecture, a new method for production of formwork for curved surfaces is robotic hot-blade cutting [3]. A heated blade or rod sweeps out a surface by melting through polystyrene foam, to create a mould for concrete casting. The shape of the blade can change dynamically during the motion, to create complex geometry; for stability and control, the blade is constrained always to form a plane curve. If a *flat* blade is used then it's not hard to see that, in order to progress smoothly, the sweeping direction needs to be orthogonal to the plane that the blade is curving in, so the blade is sweeping out planar geodesics, and the surface is a PGF. If the surface is a Monge surface, then the shape of the blade (the profile curve) remains fixed during the sweeping, and this is clearly a severe restriction. Therefore, it's interesting to ask: *can the shape of the "profile curve" change dynamically during the sweeping?* The answer to this is "no":

Theorem. *Every planar geodesic foliation can be locally parameterized as a Monge surface.*

Proof. (Outline). Given a PGF, we can assume the surface is locally parameterized as $(u,v) \mapsto \mathbf{x}(u,v)$, where the iso-curve $\gamma_{v_0}(u) := \mathbf{x}(u,v_0)$ is a planar unit-speed geodesic for each fixed v_0. Now, varying v gives us a 1-parameter family $\Pi(v)$ of planes in $\mathbb{R}^3$. Let $N(v)$ be the family of normals to these planes. By integrating $\mathbf{r}'(v) = N(v)$, with $\mathbf{r}(v_0) = \mathbf{x}(u_0,v_0)$, we obtain a curve $\mathbf{r}(v)$, the *spine curve*, that intersects the planes orthogonally. Let $(\mathbf{t}, \mathbf{q}_1, \mathbf{q}_2)$ be a minimal rotating frame along $\mathbf{r}$. Since each curve γ_v lies in the plane through $\mathbf{r}(v)$ spanned by $\mathbf{q}_i(v)$, we can write

$$\mathbf{x}(u,v) = \mathbf{r}(v) + x(u,v)\,\mathbf{q}_1(v) + y(u,v)\,\mathbf{q}_2(v)\,. \tag{2}$$

A short argument (see [2]) then shows that each iso-curve $\alpha_v(u) = (x(u,v), y(u,v))$ is a just a reparameterization of a single curve $\alpha_{v_0}(u)$. We can then reparameterize the surface so that $x(u,v)$ and $y(u,v)$ are independent of v, which is the claim of the theorem. □

Closed Spine Curves

The theorem above generates essentially all PGF's from the ingredients of a planar profile curve $\gamma_0(u)$, a space curve $\mathbf{r}(v)$, and a choice of initial vectors Q_i in the plane perpendicular to $\mathbf{r}(v_0)$. If the spine curve $\mathbf{r}(v)$ is a *closed* curve, i.e., if $\mathbf{r}(v+P) = \mathbf{r}(v)$ for some P, it does not follow that the profile curve returns to its original position after passing around the loop, to form a closed surface (see Figure 5): the other tangent direction, or *equivalently the minimal rotating frame*, of the surface needs to close up as well. This is a property of the spine curve alone.

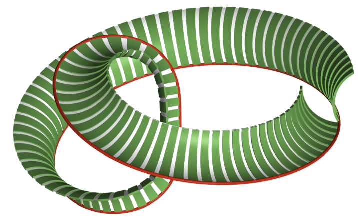

Figure 5: *A non-closing PGF surface with closed spine curve.*

In [2] we give a method to produce closed spine curves with such closed MRF's. A brief outline: assume that $\mathbf{r}(v)$ has a well-defined normal $\mathbf{n}$ and binormal $\mathbf{b}$. The Frenet frame $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ is closed (as $\mathbf{r}$ is closed) but is *not* an MRF. In fact, the *torsion* τ of the curve, given by $\mathbf{b}'(v) = \tau(v)\mathbf{n}(v)$, is the rate of rotation of the normal plane Π_v in the basis $(\mathbf{n}, \mathbf{b})$ compared with the zero rotation of the MRF. Hence, the MRF also closes up if and only if $2m\pi = \int \tau \mathrm{d}s = \int \mathbf{b}' \mathrm{d}s =$ length of $\mathbf{b}$, for some $m \in \mathbb{Z}$.

We first find a candidate periodic binormal function $\mathbf{b}$. A result of Fenchel states that a necessary and sufficient condition for a closed curve $\mathbf{b}$ in the 2-sphere to be the binormal for a closed space curve is that the *tangent geodesic curves* of the trace of $\mathbf{b}$ cover the whole sphere. It is not difficult to find such curves, and scale them so that the above closing conditions are satisfied. From there, one can solve a linear system on (for example) a space of polynomial splines to find the curvature and torsion for the desired curve $\mathbf{r}$.

If we instead take $\mathbf{b}$ with length an *odd* multiple of π, we can obtain a PGF Möbius strip, provided that the profile curve is invariant under a rotation of π. A figure-8 profile curve gives a Klein bottle, (Figure 6).

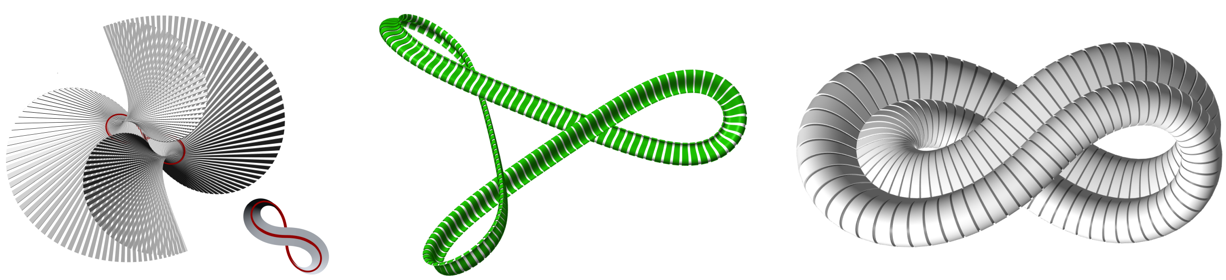

Figure 6 : *Two PGF Möbius strips (the first developable), and an immersed Klein bottle.*

Using the Methods in this Article

More details can be found in [2], and some code is available at http://geometry.compute.dtu.dk/software. For our images, we used the solutions to create wood-strip models by taking a subset of the planar geodesics and laying ruled strips along these curves. At each point on a curve, the width of the strip is proportional to the distance to the adjacent curves. Code to produce these wood-strip models is also available at the above link.

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